Anisotropic plasticity model coupled with Lode angle dependent strain-induced transformation kinetics law

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A phenomenological macroscopic plasticity model is developed for steels that exhibit strain-induced austenite-to-martensite transformation. The model makes use of a stress-state dependent transformation kinetics law that accounts for both the effects of the stress triaxiality and the Lode angle on the rate of transformation. The macroscopic strain hardening is due to nonlinear kinematic hardening as well as isotropic hardening. The latter contribution is assumed to depend on the dislocation density as well as the current martensite volume fraction. The constitutive equations are embedded in the framework of finite strain isothermal rate-independent anisotropic plasticity. Experimental data for an anisotropic austenitic stainless steel 301LN is presented for uniaxial tension, uniaxial compression, transverse plane strain tension and pure shear. The model parameters are identified using a combined analytical–numerical approach. Numerical simulations are performed of all calibration experiments and excellent agreement is observed. Moreover, we make use of experimental data from ten combined tension and shear experiments to validate the proposed constitutive model. In addition, punch and notched tension tests are performed to evaluate the model performance in structural applications with heterogeneous stress and strain fields.

1. Introduction

The stress–strain response of austenitic steels and multi-phase steels containing retained austenite is governed by both crystallographic slip and phase transformation. When subject to mechanical loading, the austenitic phase may transform into martensite, which is accompanied by an increase in volume at the microscale. The activation of the so-called TRansformation Induced Plasticity (TRIP) effect usually increases the ductility of steels as well as their strain hardening capacity (e.g., Angel, 1954). As explained by Olson and Cohen (1972), stress-induced transformation takes place when the stress level in the austenite does not exceed the deformation resistance for crystallographic slip. Here, we focus on the modeling of the elasto-plastic response of solids that undergo strain-induced transformation, which involves the plastic deformation of the austenitic phase (Olson and Cohen, 1972).
Both micromechanics-based and phenomenological macroscopic constitutive models have been developed for metastable austenitic steels. The micromechanics-based models make use of separate constitutive equations for the austenitic and martensitic phases (and possibly others) while an evolution law is used to describe the changing phase volume fractions; the effective behavior of the multi-phase material is then computed through homogenization. Several homogenization techniques have been adopted ranging from simple rules of mixtures (e.g., Lecroisey and Pineau, 1972; Hänsel et al., 1998; Hallberg et al., 2007; Santacreu et al., 2006; Post et al., 2008) to more complex homogenization techniques that account for field fluctuations within the phases (e.g., Leblond et al., 1986a, 1986b, 1989; Leblond, 1989; Stringfellow and Parks, 1991; Stringfellow et al., 1992; Bhattacharyya and Weng, 1994; Diani et al., 1995; Cherkaoui et al., 1998; Papatriantafillou et al., 2004, 2006; Turteltaub and Suiker, 2005).

The micromechanical model by Hallberg et al. (2007) for transforming solids makes use of a yield potential and a transformation potential. The transformation potential incorporates the stress state by including both the second and third invariants of the deviatoric stress, while a nonlinear rule of mixtures is employed to calculate the macroscopic flow stress of the evolving two-phase composite. Their model accounts for both the Greenwood–Johnson (Greenwood and Johnson, 1965) and the Magee (Magee, 1966) effects in a phenomenological manner. Post et al. (2008) propose a model for both stress-assisted and strain-induced martensite formation along with ageing effects. The backbone of their model is a description of the evolving dislocation densities in each phase (Estrin, 1996), assuming that the newly formed martensite will inherit the dislocation properties of the parent austenite. Leblond et al. (1986a) demonstrate through the use of Mandel-Hill homogenization (Mandel, 1964; Hill, 1967) that the macroscopic strain rate in a transforming material may be decomposed into a first term related to classical plasticity and a second related to transformation plasticity. Later, Leblond et al. (1989) developed a model that describes the transformation strain in terms of the stress deviator, the martensite content, and the martensite transformation rate for ideal-plastic phases. Leblond (1989) further expands this model by developing evolution equations for isotropic and kinematic hardening in the two phases.

Stringfellow and Parks (1991) use a self-consistent homogenization scheme to predict the inelastic stress–strain behavior of multi-phase materials, assuming isotropic and viscoplastic constituent phases. This model does not account for an evolving volume content of the phases, and its applicability is limited to materials of high rate sensitivity and small hardness differences between the constituent phases. Bhattacharyya and Weng (1994) propose an energy-based criterion to describe the constitutive behavior of ductile materials undergoing austenite-to-martensite phase transformation. Instead of assuming an explicit transformation kinetics law, the evolving martensite volume fraction is calculated incrementally from the change in Gibbs’ free energy, and the strains are estimated using the lattice parameters of the parent and transformed phases. Diani et al. (1995) use a self-consistent homogenization scheme to come up with a micromechanics-based model that describes the macroscopic transformation strain rate in terms of an effective tangent modulus, the mechanical properties of the martensite, and the rate of martensite formation.

The thermomechanical multiscale model of Turteltaub and Suiker (2005) describes the stress-induced phase transformations from a cubic to a tetragonal crystal structure. The model is developed for a single crystal of austenite, considering the kinetics and thermodynamics of transformation at multiple length scales. Papatriantafillou et al. (2006) present a constitutive model for a four-phase TRIP steel. They decompose the total strain into an elastic strain, a plastic strain, and a transformation strain, where the transformation strain contains all of the inelastic dilatation as well as a deviatoric component (modeled after Stringfellow et al., 1992). They develop hardening laws for the individual phases and use a homogenization technique for nonlinear composites (Ponte Castañeda, 1991, 1992; Suquet, 1996) to estimate the strain distribution in the individual phases as well as the resulting macroscopic properties of the material.

In phenomenological macroscopic models, the constitutive equations are no longer given for individual phases and are directly formulated at the macroscopic level. Motivated in part by the experimentally observed lower equivalent stress under torsion as compared with compression (Miller and McDowell, 1996a), Miller and McDowell (1996b) propose a macroscopic plasticity model that incorporates the third invariant of the overstress tensor, where overstress is defined as the difference between the Cauchy stress and the back stress. They conclude that the difference in strain hardening behavior is due to both geometric effects, as shear deformation results in a lower Taylor factor than compression, as well as stress-state dependent material hardening, as there are more possible planes of maximum shear stress in compression than torsion. Cherkaoui et al. (1998) develop a thermomechanical model to describe the behavior of a single crystal of austenite undergoing strain-induced phase transformation. They propose a specific form of the Helmholtz free energy and make use of the corresponding thermodynamic driving forces to model the phase transformation and plastic flow.

Mohr and Jacquemin (2008) proposed a macroscopic plasticity model that makes use of a Hill’48 (Hill, 1948) yield surface along with a non-associated hardening model to describe the direction dependency of the strain hardening in an anisotropic stainless steel of type 301LN. Hänsel et al. (1998) developed a temperature-dependent isotropic constitutive model for TRIP steels. It makes use of the isothermal transformation kinetics law proposed by Tsuta and Cortes (1993), in which the martensite volume content is defined as a function of the von Mises equivalent plastic strain. Hänsel et al. (1998) introduce an empirical weighting function of temperature to account for the temperature-dependency of the phase transformation. The macroscopic behavior is described through a von Mises yield surface with an isotropic hardening law that depends both on the equivalent plastic strain and the martensite content. Schedin et al. (2004) slightly modified the implementation of the model by Hänsel et al. (1998) through the introduction of an anisotropic yield function. This empirical model is implemented in the commercial finite element software LS-DYNA, but does not account for the effect of stress-state on martensite evolution.
Many experimentalists have shown that the phase transformation in austenitic steels is stress-state dependent (e.g., Cina, 1954; Powell et al., 1958; Hecker et al., 1982; Murr et al., 1982; Young, 1988; Kosarchuk et al., 1989; Okutani et al., 1995; DeMania, 1995; Miller and McDowell, 1996a; Iwamoto et al., 1998; Lebedev and Kosarchuk, 2000; Shin et al., 2003; Yan et al., 2006; Perdahcioglu et al., 2008; Nanga et al., 2009). In other words, the transformation kinetics cannot be described as a function of the von Mises equivalent plastic strain only. Stringfellow et al. (1992) addressed this issue by incorporating the effect of stress-triaxiality into their micromechanics-based plasticity model. However, the recent experimental analysis of Beese and Mohr (2011b) has explicitly shown that the third invariant of the deviatoric Cauchy stress tensor (Lode angle) affects the rate of phase transformation under isothermal conditions in addition to the stress triaxiality.

It is the objective of the present paper to develop an isothermal rate-independent phenomenological macroscopic finite strain plasticity model for solids undergoing strain-induced austenite-to-martensitic transformation. Different from state-of-the-art models, the proposed model will incorporate an isotropic hardening model that is coupled with a stress-state dependent transformation kinetics law. The proposed constitutive model is composed of an anisotropic yield function with a nonlinear kinematic hardening law of the Frederick–Armstrong type (e.g., Armstrong and Frederick, 1966, 2007; Chaboche, 1977, 2008), and an isotropic hardening function that is coupled with a stress-state dependent transformation kinetics law. The model is implemented into a finite element software, calibrated based on selected experiments, and used to predict the stress–strain response for various loading conditions. It is found that the proposed model is able to describe the material’s constitutive response over a wide range of stress states including uniaxial tension, uniaxial compression, transverse plane strain tension, simple shear and equibiaxial tension.

2. Experimental procedures and results

The results from static experiments performed by Beese and Mohr (2011b) on a stainless steel 301LN (SS301LN) are briefly summarized. In particular, we focus on the experiments for uniaxial tension, in-plane uniaxial compression, shear, and equibiaxial tension. During each experiment, the applied force and the displacement fields on the specimen surface are measured using Digital Image Correlation (DIC). In addition, a ferritescope is used to measure the martensite evolution as a function of the strain and stress state (Beese and Mohr, 2011a).

2.1. Material

All specimens have been extracted from 1.5 mm thick temper-rolled austenitic stainless steel sheets of the specification 18-7L (AISI classification 301LN) provided by ArcelorMittal. This material is a metastable austenitic steel, whose initial microstructure is composed of a face-centered cubic (FCC) γ-austenite, body-centered cubic (BCC) α’-martensite, and a small volume fraction (less than 5%) of hexagonally close-packed (HCP) ε-martensite. The chemical composition of the sheets is: 17.5% chromium, 6.5% nickel, 0.025% carbon and 0.15% nitrogen. The sheet material is anisotropic, with Lankford ratios of $r_0 = 0.67$, $r_{45} = 0.67$, and $r_{90} = 0.89$ for uniaxial tension along the rolling, 45°, and cross-rolling directions, respectively.

2.2. Uniaxial loading

Static uniaxial tension and in-plane uniaxial compression tests are performed. During both types of experiments, the gage section is painted white with a black speckle pattern and DIC is used to determine the displacement and strain fields on the specimen gage section surfaces. The uniaxial tension experiments are standard and adhere to ASTM E8. However, the in-plane compression tests require a custom-made anti-buckling device to prolong the initial range of membrane-dominated deformation. As detailed in Beese and Mohr (2011a), this device is composed of two 6.35 mm thick aluminum plates, which sandwich the gage section of the in-plane compression specimen. A set of 14 springs holds the two face plates together by applying a compressive pressure of about 1.5 MPa. This pressure is sufficiently high to delay the transition from membrane- to bending-dominated loading due to buckling. At the same time, it is sufficiently low such that the measured stress–strain curve is not affected by the friction between the specimen and the aluminum plates. Teflon between the face plates and the gage section renders the frictional effects negligible. Using this procedure, the uniaxial stress–strain curve for compression could be determined for true strains of up to 10%. The measured stress–strain curves for uniaxial tension and compression are shown in Fig. 1.

2.3. Combined tension and shear

Mohr and Jacquemin (2008) performed combined tension and shear experiments on the present sheet material. They used a custom-made hydraulic dual-actuator loading machine and follow the experimental procedures developed by Mohr and Oswald (2008). The specimens used are flat sheet specimens with a reduced-thickness gage section (Fig. 2), resulting in plane strain conditions along the horizontal direction of the specimen and plane stress through the gage thickness of the specimen. The tests are performed under force-control, and various combinations of shear and tensile loading are applied to each specimen by varying the so-called biaxial loading angle $\beta$ (Fig. 2), which describes the ratio of the vertical (normal)
force to the horizontal (shear) force. Here, we make use of their results from a first series of experiments where the horizontal axis (plane strain direction) is aligned with the material rolling direction, and a second series where the horizontal axis coincides with the cross rolling direction.

3. Plasticity model

A phenomenological model is developed to describe the large deformation behavior of the austenitic stainless steel under static loading at room temperature. In addition to the equivalent plastic strain, we introduce the martensite content as internal state variable because of its first order effect on the rate of strain hardening. In the following, we outline the rate independent finite strain constitutive equations, which involve the yield surface, flow rule, isotropic hardening law, kinematic hardening law, and the martensite transformation kinetics law. Bold upper case letters (e.g. $B$) and double-underscored lowercase bold letters (e.g. $b$) are used to denote matrices and tensors, while bold lowercase letters without underscore (e.g. $b$) are used to denote vectors. Square brackets are exclusively used to indicate the arguments of a function, while round and curly brackets are employed to signify the precedence of mathematical operations.

\[1\text{ It is noted that the equivalent plastic strain may not be interpreted as a measure of shape change (unlike the plastic strain tensor and its invariants).}\]
3.1. Kinematic of finite strain

The constitutive model is implemented in the commercial finite element software Abaqus/explicit. Therefore, the standard finite strain formulation for plane stress shell elements with co-rotational coordinate frames is used (Abaqus, 2008). The Cauchy stress tensor in the current configuration is denoted as $\bar{\sigma}$, while $d\bar{\varepsilon}$ denotes the work-conjugate strain increment. Stress and strain components, $\sigma_{ij}$ and $\varepsilon_{ij}$, are reported in the current material coordinate systems, assuming that the orthotropic material symmetry is preserved throughout loading. Formally, we write

$$\bar{\sigma} = \sigma_{ij}(Re_{i} \otimes Re_{j}),$$  

with $R$ denoting the rotation of the co-rotational material coordinate system; the unit vectors $e_{1}$ and $e_{2}$ are aligned with the initial rolling and cross-rolling directions.

3.2. Yield surface

The results from multi-axial experiments (Mohr and Jacquemin, 2008) have demonstrated that the quadratic anisotropic yield function by Hill (1948) provides a good approximation of the initial yield surface of the stainless steel 301LN sheet material. It is defined as

$$f = \bar{\sigma} - k = 0,$$  

where $\bar{\sigma} = \sigma_{[ij]}$ defines the equivalent stress, $k$ is the deformation resistance, and $\bar{\sigma}$ is the Cauchy stress tensor. The Hill’48 equivalent stress is typically given in the form

$$\bar{\sigma} = \sqrt{F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2 + 2L\sigma_{12}^2 + 2M\sigma_{13}^2 + 2N\sigma_{23}^2},$$  

where $F$, $G$, $H$, $L$, $M$, and $N$ are the coefficients describing the material anisotropy. In the present model, we introduce a deviatoric back stress tensor $a$, and replace the components $\sigma_{ij}$ in Eq. (3) by $\sigma_{ij} - a_{ij}$.

3.3. Associated flow rule

An associated flow rule is chosen to describe the evolution of the plastic strain tensor. Therefore, the increment in plastic strains, $d\bar{\varepsilon}^p$, is proportional to the derivative of the equivalent stress,

$$d\bar{\varepsilon}^p = d\lambda \frac{\partial \bar{\varepsilon}}{\partial \bar{\sigma}} a$$  

where $d\lambda \geq 0$ is the plastic multiplier. The integral $\tau_p = \int d\lambda$ is referred to as anisotropic equivalent plastic strain.

3.4. Kinematic hardening

The nonlinear kinematic hardening rule is written as

$$d\bar{\sigma} = c_L d\lambda \sigma^p - c_N L d\lambda,$$  

where $c_L$ and $c_N$ are material parameters. For $c_NL = 0$, Eq. (5) reduces to the linear kinematic hardening law by Prager (1949); in this case, the back stress evolution is unbounded and converges towards a saturation value under monotonic loading. As discussed by Lemaître and Chaboche (1994), the dynamic recovery term may be interpreted as a description of the “fading memory effect of the strain path.” For example, in the case of uniaxial tension, the back stress evolution asymptotically approaches a saturation value.

3.5. Isotropic hardening

In addition to a kinematic hardening law, an isotropic hardening law is used to describe the evolution of the deformation resistance $k$ during plastic loading. It is assumed that the deformation resistance depends on the evolution of the equivalent plastic strain and the martensite volume fraction. We decouple the mechanisms of increased deformation resistance due to dislocation pile-up from that due to the formation of martensite in an additive manner. We write

$$dk = H_e d\bar{\varepsilon}^p_{vM} + H_f d\gamma,$$  

where $d\bar{\varepsilon}^p_{vM}$ is the increment in the isotropic equivalent plastic strain (von Mises equivalent plastic strain), and $d\gamma$ is the increment in the martensite volume fraction. Under the absence of phase transformation, it is assumed that the deformation resistance is an exponential function of the von Mises equivalent plastic strain

$$k = k_0 + H_0 \{1 - \exp[-A\bar{\varepsilon}^p_{vM}]\},$$  

where $H_0$, $A$, and $k_0$ are material parameters.
with the model parameters $k_0$, $H_0$, and $A$. Consequently, the relationship for the isotropic strain hardening modulus reads

$$H_0 \left[ \psi^{\rho}_{VM} \right] = \frac{dk}{d \rho} = AH_0 \exp \left[ -A \psi^{\rho}_{VM} \right]$$  \hspace{1cm} (8)$$

The second hardening modulus, $H$, characterizes the rate of strain hardening due to an increasing martensite volume fraction, and is assumed to be constant. It is noted that the yield surface shape changes are considered as a second order effect and are neglected by the present constitutive model. The evolution of texture is certainly important when very large strains are present such as during cold-rolling (see e.g. Raabe, 1997).

### 3.6. Transformation kinetics law

The isotropic transformation kinetics law developed in Beese and Mohr (2011b) is used to describe the martensite evolution as a function of the equivalent plastic strain and the stress state. The latter will be characterized by the stress triaxiality and the Lode angle parameter. The stress triaxiality $\eta$ is proportional to the ratio of the first invariant of the Cauchy stress tensor, $I_1$, and the second invariant of the deviatoric stress tensor, $J_2$.

$$\eta = \frac{\sigma_m}{\sigma_{VM}} \text{ with } \sigma_m = \frac{tr \left[ \sigma \right]}{3} = \frac{I_1}{3} \text{ and } \sigma_{VM} = \sqrt{3J_2} = \sqrt{\frac{2}{3} \bar{s} \cdot \bar{s}}$$  \hspace{1cm} (9)$$

where $\bar{s}$ is the deviatoric stress tensor. The dimensionless Lode angle parameter, $\bar{\theta}$, depends on the ratio of the second and third invariants of the deviatoric stress tensor, $J_2$ and $J_3$. Its definition reads

$$\bar{\theta} = 1 - \frac{2}{\bar{r}} \arccos \left[ \frac{3 \sqrt{3} J_3}{2 \sqrt{J_2}} \right] \text{ with } J_3 := \det \left[ \bar{s} \right]$$  \hspace{1cm} (10)$$

According to the transformation kinetics law by Beese and Mohr (2011b), the change of martensite volume fraction is governed by the differential equation

$$\frac{d \chi}{d \lambda} = (\chi_{max} - \chi) m D (d \psi^{\rho}_{VM})^{-1} (d \psi^{\rho}_{VM})$$  \hspace{1cm} (11)$$

The function $D$ depends on both the stress triaxiality and the Lode angle parameter,

$$D = (D_0 + a_\eta \eta + a_\sigma \sigma) +$$  \hspace{1cm} (12)$$

with the material model parameters $\chi_{max}$, $m \geq 0$, $D_0$, $a_\eta$, and $a_\sigma$. The maximum operator $z_+ = \max(z, 0)$ is used to ensure that $D \geq 0$ for any $\eta$ and $\bar{\theta}$. In the special case of $a_\eta = a_\sigma = 0$, the above relationship reduces to the transformation kinetics law of Santacreu et al. (2006) for isothermal conditions.

### 3.7. Complementary conditions

The direction of plastic evolution is unilaterally constrained through the Kuhn–Tucker complementary conditions
d$\lambda \geq 0$, $df \leq 0$ and $(d \lambda)(df) = 0$  \hspace{1cm} (13)$$

Consequently, the isotropic strain hardening is irreversible. The evolution direction of the martensite content is also implicitly constrained to $d \lambda \geq 0$ by the particular choice of the transformation kinetics law. Thus, we only allow for a transformation from austenite-to-martensite, while the reverse transformation is prohibited.

### 3.8. Specialization for plane stress

For the case of plane stress, it is worth rewriting the constitutive equations in vector notation. We define the stress vector

$$\sigma = \left[ \begin{array}{c} \sigma_0 \\ \sigma_{90} \\ \tau \end{array} \right]$$  \hspace{1cm} (14)$$

with the Cauchy stress components $\sigma_0 = \sigma_{11}$, $\sigma_{90} = \sigma_{22}$ and $\tau = \sigma_{12}$. The strain vector definition reads

$$\epsilon = \left[ \begin{array}{c} \epsilon_0 \\ \epsilon_{90} \\ \gamma \end{array} \right]$$  \hspace{1cm} (15)$$

where $\epsilon_0 = \epsilon_{11}$ and $\epsilon_{90} = \epsilon_{22}$ are the logarithmic strains along the rolling and cross-rolling directions; $\gamma = 2 \epsilon_{12}$ denotes the corresponding in-plane shear strain (which is twice the mathematical shear strain). The yield function with back stress for plane stress conditions reduces to

$$\sigma = \sqrt{F(\sigma_{22} - \sigma_{22} + \sigma_{33})^2 + G(\sigma_{11} - \sigma_{11} + \sigma_{33})^2 + H(\sigma_{11} - \sigma_{11} - \sigma_{22} + \sigma_{33})^2 + 2L(\sigma_{22} - \sigma_{12})^2}$$  \hspace{1cm} (16)$$
We introduce the back stress vector $\mathbf{b}$ with the components $b_1 = a_{11}/C_0 a_{33}$, $b_2 = a_{22}/C_0 a_{33}$ and $b_3 = a_{12}$, and rewrite Eq. (16) as

$$\mathbf{\sigma} = \sqrt{\mathbf{P}(\mathbf{\sigma} - \mathbf{b}) \cdot (\mathbf{\sigma} - \mathbf{b})}$$

(17)

with the anisotropy coefficient matrix

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} = \begin{bmatrix} G+H & -H & 0 \\ -H & F+H & 0 \\ 0 & 0 & 2L \end{bmatrix}$$

(18)

The corresponding flow rule for plane stress reads

$$d\mathbf{e}^p = \frac{d\mathbf{l}}{\mathbf{G} \cdot (\mathbf{\sigma} - \mathbf{b})}$$

(19)

while the thickness strain is computed based on the assumption of plastic incompressibility, $d\mathbf{e}^p_{33} = -(d\mathbf{e}^p_{0} + d\mathbf{e}^p_{90})$.

The equations for isotropic hardening remain unchanged; however, the nonlinear kinematic hardening rule must be revisited. Evaluation of the kinematic hardening rule (5) for plane stress yields the evolution equation

$$d\mathbf{b} = c_L \mathbf{G} d\mathbf{e}^p - c_{NL} \mathbf{b} \mathbf{d} = \left\{ \frac{c_L}{\sigma} \mathbf{G} (\mathbf{\sigma} - \mathbf{b}) - c_{NL} \mathbf{b} \right\} d\lambda$$

(20)

where

$$\mathbf{G} = \frac{2}{3} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

(21)

4. Model calibration and validation

The above anisotropic plasticity model for metastable austenitic steel sheets requires the identification of seventeen model parameters:

- Yield surface coefficients: $P_{11}$, $P_{12}$, and $P_{33}$
- Isotropic hardening: $k_0$, $A$, $H_e$, and $H_w$
- Kinematic hardening: $c_L$ and $c_{NL}$
- Initial back stress: $b_0^1$, $b_0^2$, and $b_0^3$
- Transformation kinetics: $\lambda_{\text{max}}$, $m$, $D_0$, $a_0$, and $a_0$

All material model parameters may be identified based on experiments along the rolling and cross-rolling directions for

1. uniaxial tension,
2. uniaxial compression,
3. transverse plane strain tension.

In addition, the results from an experiment for

4. shear loading with $\alpha = 0^\circ$.

is used for calibration. The loading conditions in the latter experiment correspond to pure shear loading in the machine coordinate system (vertical force is zero, see Mohr and Jacquemin, 2008). Note that the material frame is aligned with the machine coordinate system at the beginning of the shear experiment; as a result, the experimental results for $\alpha = 0^\circ$ and $\alpha = 90^\circ$ are almost identical for shear loading.

4.1. Model parameter identification

The identification of the material model parameters associated with the transformation kinetics has been detailed in Beese and Mohr (2011b) and will not be discussed here. The remaining model parameters are identified in a two-step procedure: a first estimate of the material model parameters is made based on approximate analytical solutions of the constitutive equations for simple loading conditions; subsequently, the analytical estimates are used as seed values for the parameter identification through a Nelder–Mead Simplex algorithm (Nelder and Mead, 1965). The computational version of the plane stress constitutive model is used for the latter step.

Initial values for the anisotropy matrix components $P_{11}$, $P_{12}$, and $P_{33}$ are obtained from the measured Lankford parameters $r_0 = 0.67$, $r_{45} = 0.67$, and $r_{90} = 0.89$. With $\alpha$ denoting the angle between the tensile direction and the rolling
direction, we have

\[ r_x = \frac{H + (2L - F - G - 4H)\sin^2 \alpha \cos^2 \alpha}{F \sin^2 \alpha + G \cos^2 \alpha} \tag{22} \]

and thus \( P_1 = 1.17, P_{12} = -0.47, \) and \( P_{33} = 2.88. \)

The measured initial yield stresses for tension and compression are used to identify the seed values for the initial back stresses \( \beta_1^0, \beta_2^0, \) and \( \beta_3^0 \) and the initial deformation resistance \( k_0. \) Before determining the initial back stresses, it is worth looking at their expected development throughout temper-rolling. For this, we rewrite Eq. (20) in terms of the increments in plastic strain as

\[ d\beta = \frac{2}{3} C_l \left[ \begin{array}{c} 2d\epsilon_0^p + d\epsilon_90^p \\ d\sigma_0^p + 2d\epsilon_90^p \\ \frac{1}{2}d\tau^p \end{array} \right] \]

\[ -c_{\text{NL}} \hat{\alpha} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \tag{23} \]

Throughout rolling, we have \( d\gamma^p = 0 \) (based on the assumption of a homogeneous strain distribution though the sheet thickness) and we therefore assume \( \beta_3^0 = 0. \)

After (back-)extrapolating the slope of the respective stress–strain curves at a plastic strain of 0.02 to zero plastic strain, we obtain the initial yield stresses \( \sigma_{y,0,T} = 830 \text{ MPa}, \) \( \sigma_{y,90,T} = 890 \text{ MPa}, \) \( \sigma_{y,0,C} = -900 \text{ MPa}, \) and \( \sigma_{y,90,C} = -1100 \text{ MPa}. \) The corresponding analytical solutions for uniaxial tension read

\[ \sigma_{11} = \sigma_0 = \beta_1 + \frac{P_{12}}{P_{11}} \beta_2 + \sqrt{\beta_1^2 \left( \frac{P_{12}^2}{P_{11}^2} - \frac{P_{22}}{P_{11}} \right) - \frac{P_{33}}{P_{11}} \beta_3^2 + \frac{k^2}{P_{11}^2}} \tag{24} \]

and

\[ \sigma_{22} = \sigma_{90} = \frac{P_{12}}{P_{22}} \beta_1 + \beta_2 + \sqrt{\beta_1^2 \left( \frac{P_{12}}{P_{22}} \right)^2 - \frac{P_{11}}{P_{22}} \left( \frac{P_{22}}{P_{11}} \right)^2 - \frac{P_{33}}{P_{22}} \beta_3^2 + \frac{k^2}{P_{22}^2}} \tag{25} \]

For uniaxial compression, we find

\[ \sigma_{11} = \sigma_0 = \beta_1 + \frac{P_{12}}{P_{11}} \beta_2 + \sqrt{\beta_1^2 \left( \frac{P_{12}}{P_{11}} \right)^2 - \frac{P_{22}}{P_{11}} \left( \frac{P_{22}}{P_{11}} \right)^2 - \frac{P_{33}}{P_{11}} \beta_3^2 + \frac{k^2}{P_{11}^2}} \tag{26} \]

and

\[ \sigma_{22} = \sigma_{90} = \frac{P_{12}}{P_{22}} \beta_1 + \beta_2 + \sqrt{\beta_1^2 \left( \frac{P_{12}}{P_{22}} \right)^2 - \frac{P_{11}}{P_{22}} \left( \frac{P_{22}}{P_{11}} \right)^2 - \frac{P_{33}}{P_{22}} \beta_3^2 + \frac{k^2}{P_{22}^2}} \tag{27} \]

After evaluating the above four equations, we use \( \beta_1^0 \approx -95 \text{ MPa}, \) \( \beta_2^0 \approx -150 \text{ MPa}, \) and \( k_0 \approx 990 \text{ MPa} \) as seed values for the numerical optimization.

The stress–strain response for uniaxial tension in the cross direction is compared with that for uniaxial compression along the rolling direction to obtain a first estimate of the isotropic hardening parameters. The former loading state produces the highest rate of martensite evolution (with respect to axial strain), while the latter produces the lowest (Beese and Mohr, 2011b). For example, at axial strains of about 15% and 11%, respectively, we observe

- \( \sigma_{90} \big|_{\text{ax}} = 0.15 \) = 1294 MPa and \( \chi = 85\% \) for tension along the cross-direction;
- \( \sigma_0 \big|_{\text{ax}} = 0.11 \) = -1079 MPa and \( \chi = 35\% \) for compression along the rolling direction;

Using an estimated value of \( A = 5, \) we obtain the moduli \( H_0 \approx 70 \text{ MPa} \) and \( H_1 \approx 400 \text{ MPa} \) from the above results. The values \( C_1 = 750 \text{ MPa} \) and \( C_{11} = 2 \) are used as initial guesses for the kinematic hardening parameters.

The final set of parameters after computational optimization is given in Table 1. Note that the parameters \( M \) and \( N \) were set to 1.5, which corresponds to the value in the case of an isotropic material. Out-of-plane shear stresses develop in the post-necking regime of a notched tensile experiment. We therefore reran the corresponding simulations for \( M = N = 1 \) and \( M = N = 2 \) and found almost the same force–displacement response as for \( M = N = 1.5. \) It is therefore tentatively concluded that they have a negligible effect.

4.2. Comparison of numerical and experimental results

The simulation results of the calibrated model agree well with the experimentally measured stress–strain curves for all seven experiments used for calibration. For uniaxial tension, the maximum difference between the simulation and the experimental result does not exceed 7% of the stress level (Fig. 3). The wide spread in stress level for uniaxial compression along the rolling and cross-rolling directions is predicted correctly by the model (Fig. 4). Furthermore, the good agreement of the
Table 1
Calibrated material parameters for 1.5 mm-thick temper-rolled stainless steel 301LN sheets compared to initial seed values.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Seed value</th>
<th>Final value</th>
<th>Ratio of final to seed value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anisotropy coefficients</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_{11}$</td>
<td>1.17</td>
<td>1.16</td>
<td>0.99</td>
</tr>
<tr>
<td>$P_{12}$</td>
<td>-0.47</td>
<td>-0.552</td>
<td>1.17</td>
</tr>
<tr>
<td>$P_{13}$</td>
<td>2.88</td>
<td>3.2</td>
<td>1.11</td>
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<td>Isotropic hardening parameters</td>
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<tr>
<td>$k_0$ (MPa)</td>
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<td>962.9</td>
<td>0.97</td>
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<tr>
<td>$A$</td>
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<td>5.6</td>
<td>1.12</td>
</tr>
<tr>
<td>$H_e$ (MPa)</td>
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<td>103.1</td>
<td>1.47</td>
</tr>
<tr>
<td>$H_w$ (MPa)</td>
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<td>396.4</td>
<td>0.99</td>
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<td>Kinematic hardening parameters</td>
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<td></td>
</tr>
<tr>
<td>$c_{01}$ (MPa)</td>
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<td>766.9</td>
<td>1.02</td>
</tr>
<tr>
<td>$c_{02}$ (MPa)</td>
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<td>2.029</td>
<td>1.01</td>
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<td>$\beta_0^1$ (MPa)</td>
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<td>$\beta_0^2$ (MPa)</td>
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<td>$m$</td>
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<td>$D_0$</td>
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</tr>
<tr>
<td>$a_1$</td>
<td>2</td>
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</table>

Fig. 3. Experimental stress–plastic strain curves (symbols) and stress–strain curves described by model (solid lines) for uniaxial tension in the rolling (red), 45° (black), and cross-rolling (blue) directions. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 4. Compressive axial plastic strain versus compressive axial stress under uniaxial compression: model prediction (solid lines) versus experimental data (symbols).
results shown in Fig. 5 demonstrates the model's ability to describe the pronounced difference between the rate of strain hardening for uniaxial tension and compression (recall from Fig. 1). The results for the remaining three calibration experiments (plane strain tension and pure shear) are summarized in Fig. 6. Again, we observe a satisfactory agreement between the model and experimental results, which is mostly due to the adequate modeling of the effect of stress-state on the rate of strain hardening. The plastic width strain is shown as a function of the plastic thickness strain in Fig. 7 for uniaxial tension. Good agreement is observed for tension along the rolling, cross-rolling and diagonal directions. Additional experiments need to be performed in the future to confirm the assumption of associated plastic flow for intermediate loading directions.

We also performed numerical simulations of the combined tension and shear experiments by Mohr and Jacquemin (2008). The corresponding model predictions (solid lines) are shown next to the experimental data (dashed lines) in Figs. 8 and 9. We observed good agreement (differences in stress level of less than 30 MPa between experiments and simulations) for all shear stress–strain curves. Among the normal stress–strain curves, we observe good agreement for $\alpha = 0^\circ$ and $\alpha = 90^\circ$, while the largest difference is observed at large strains for transverse plane strain tension ($\beta = 90^\circ$) along the diagonal direction ($\alpha = \pm 45^\circ$).

4.3. Comment on the model of Mohr and Jacquemin

Mohr and Jacquemin (2008) concluded that the hardening behavior of the present austenitic stainless steel sheet material could not be described by conventional isotropic-kinematic hardening models. They proposed a non-associated hardening model where the self-similar evolution of a Hill’48 yield surface is a function of an anisotropic equivalent plastic strain that is different from the associated Hill’48 equivalent plastic strain. This proposition was made based on the speculation that the rate of phase transformation is direction dependent. Note that Mohr and Jacquemin (2008) did not have any experimental data on the martensite content evolution and built their model based on macroscopic observations only. We recently developed a reliable experimental procedure for measuring the martensite content (Beese and Mohr, 2011a). It turns out that our experimental data
does not support the modeling assumptions made by Mohr and Jacquemin (2008). Based on our experimental observations, we therefore developed an entirely new model which explains the different hardening curves not through anisotropy, but stress-state dependency of the transformation kinetics. The accuracy of the new model's stress–strain curve predictions is similar to that of

**Fig. 6.** Top: Plastic axial strain versus axial stress under plane strain tension: model prediction (solid lines) versus experimental data (open symbols). Equivalent plastic strain (von Mises definition) versus von Mises equivalent stress under pure shear: model prediction (solid lines) versus experimental data (open symbols). Bottom: Predicted martensite formation under plane strain tension (dashed green line) and under pure shear (dotted gray line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

**Fig. 7.** Plastic width strain versus plastic through-thickness strain (R-ratio) under uniaxial tension in the rolling, 45°, and cross-rolling directions up to an axial strain of 0.2 as found in experiment (open symbols) and calculated by the model (solid lines).
the model of Mohr and Jacquemin, but the underlying model assumptions regarding the origin of the observed non-conventional hardening behavior are now consistent with the identified austenite-to-martensite transformation kinetics.

5. Structural validation

Two simple structural experiments are chosen to demonstrate the performance of the constitutive model in applications with heterogeneous stress and strain fields. A punch test is carried out where a clamped sheet is subject to out-of-plane loading. Furthermore, a notched tensile test is performed under quasi-static loading conditions. Simulations are performed of each experiment and compared with the experimental results.

5.1. Punch loading

A circular disk specimen with a diameter of 126 mm is extracted from the stainless steel sheets and clamped onto a 49.2 mm diameter die. A set of sixteen ½”-20 bolts is used to apply the clamping pressure. The specimen is then loaded using a hemispherical punch with a radius of 22.2 mm. Five sheets of 0.05 mm-thick Teflon with grease are placed between the specimen and the punch to reduce frictional effects. The experiment is performed on an electro-mechanical testing machine under displacement control. The force is measured using a 200 kN load cell, while an LVDT is used to measure the crosshead displacement. The effective punch displacement is determined from the crosshead displacement assuming an overall machine stiffness of 100 kN/mm. A representative force versus punch displacement curve is shown as dashed line in Fig. 10. After a soft bending-dominated initial response, we observe a monotonically increasing force–displacement curve until a crack forms at the apex of the punched specimen.

In the corresponding computational model both the die and punch are represented by analytical rigid surfaces. We take advantage of symmetry in the mechanical system and model only one quarter of the disk specimen using four-node reduced-integration plane stress shell elements (element type S4R from the Abaqus library). The elements have a side length of 1 mm and feature five thickness integration points. A low friction coefficient of 0.04 is assumed between the punch and specimen, while a high friction coefficient of 0.5 is used between the die and the specimen in the clamping area. The simulations are performed using about 100,000 explicit time steps. The predicted force–displacement curve is shown as a solid line in Fig. 10. The comparison with the experimentally measured curve demonstrates the model’s ability to predict the force–displacement response in a punch experiment with good accuracy.
5.2. Notched tension

Notched tension is another important structural experiment, which involves a range of stress states that are frequently encountered in sheet metal forming. The geometry of the notched tensile specimen is shown in Fig. 11(a). The circular cutouts have a radius of 6.67 mm while the minimum width at the notch is 10 mm. A DIC-based 34 mm long virtual extensometer is used to measure the relative displacement of the specimen shoulders. The recorded force–displacement is shown as a dotted line in Fig. 11(b). It features a force maximum at a relative displacement of about 1.6 mm. Localized necking through the thickness becomes dominant beyond this point which explains the decrease in force before the point of fracture is reached.
The mesh of a quarter of the specimen is shown in Fig. 11(b). It features 3200 reduced-integration shell elements (type S4R). The elements in the notch region have a side length of 0.125 mm, while those in the shoulder region have a maximum side length of 0.525 mm. We apply symmetry boundary conditions to the horizontal and vertical axes of symmetry, and a constant vertical velocity to all nodes at the top boundary of the specimen. A total of 100,000 explicit time steps are performed to simulate the static experiment. The computed force–displacement curve is compared with that obtained experimentally. We note that there is excellent agreement between the two curves up to the point of maximum force. Beyond this point, the validity of plane stress assumption of the shell element formulation breaks down as out-of-plane stresses develop throughout necking.

To overcome this limitation associated with the plane stress assumption, we developed a time integration scheme for the three-dimensional constitutive model and programmed a corresponding user material subroutine. Simulations are subsequently performed with eight-node first-order solid elements (type C3D8R of Abaqus element library) through the thickness direction of a mesh of an eighth of the specimen (exploiting the symmetry with respect to the sheet mid-plane). The corresponding force–displacement curve (dashed curve in Fig. 11b) coincides with that for the shell element model up to the force maximum, and continues in close agreement with the experimental curve up to the point of fracture. The contours of the anisotropic equivalent plastic strain and martensite content are shown in Fig. 11(c), while the evolutions of martensite volume fraction, equivalent plastic strain, stress triaxiality, and Lode angle parameter are shown in Fig. 11(d). Observe that the material has fully transformed into martensite before fracture occurs.

6. Conclusions

A phenomenological macroscopic constitutive model is proposed for steels undergoing deformation-induced austenite-to-martensite phase transformation. Several experimental studies on the evolution of the martensite content in austenitic
stainless steels (e.g., Cina, 1954; Powell et al., 1958; Kosarchuk et al., 1989; DeMania, 1995; Iwamoto et al., 1998) have demonstrated that the martensitic transformation is stress-state dependent. Here, we make use of a newly developed transformation kinetics law that accounts for the stress triaxiality as well as the Lode angle parameter to describe the stress-state dependency. After developing the corresponding time integration scheme and implementing the model into a nonlinear explicit finite element code, the material model parameters are calibrated based on isothermal static experiments for uniaxial tension, transverse plane strain tension, uniaxial compression and pure shear. It is subsequently validated for combined tension and shear experiments and two structural experiments. Good agreement of the model predictions and the measured force–displacement curves is observed for all calibration and validation experiments. In particular, the stress-state dependency of the strain hardening is captured by the model due to the coupling with an advanced transformation kinetics law.

Acknowledgments

Thanks are due to Professor David Parks (MIT) and Dr. Pierre-Olivier Santacreu (ArcelorMittal) for valuable discussions. The partial financial support of the joint MIT/Industry Advanced High Strength Steels (AHSS) Fracture Consortium and Aperam (former ArcelorMittal, Isbuergues, France) is gratefully acknowledged.

Appendix A. Computational model

The time integration scheme is developed to implement the proposed constitutive model for plane stress into a nonlinear explicit finite element code. For a given strain increment \( \Delta e = e_{n+1} - e_n \), the algorithm provides an approximate solution to the strain-driven problem, where the state variables \( \alpha_{n+1} \) and \( \lambda_{n+1} \), the equivalent strains \( e_{p,n+1} \) and \( e_{p,n+1}^{M,n+1} \), and stresses \( \sigma_{n+1} \) are computed at time \( t_{n+1} = t_n + \Delta t \) based on the solution at time \( t_n \). If the elastic trial stress

\[
\sigma^{tr} = C(e_{n+1} - e_n^p)
\]

lies outside the elastic domain,

\[
f[\sigma^{tr}, \alpha_n, e_n, \lambda_n] > 0,
\]

a return-mapping scheme is employed to solve the constitutive equations. The equivalent plastic strain, \( \lambda \), is used as an evolution parameter to discretize all evolution equations. A backward-Euler (implicit) integration procedure is used to approximate the derivatives with respect to \( \lambda \),

\[
y_{n+1} = y_n + \Delta \lambda \frac{dy}{d\lambda} \bigg|_{n+1}
\]

and thus

\[
\frac{\partial y_{n+1}}{\partial (\Delta \lambda)} = \frac{dy}{d\lambda} \bigg|_{n+1}
\]

In order to determine the stress and state variables at time \( t_{n+1} \), we solve the discretized consistency condition at time \( t_{n+1} \),

\[
\frac{df}{d\lambda} \bigg|_{n+1} = f_{n+1} - f_n = 0
\]

For sufficiently small time steps and smooth loading histories, we have \( f_n \approx 0 \). Thus, the computational problem reduces to determining \( \Delta \lambda \) such that

\[
f_{n+1} = f_{n+1} = 0
\]

An iterative Newton–Raphson scheme is employed to solve (33). During the ith iteration, the solution \( \Delta \lambda_{i+1} \) is obtained from linearizing Eq. (33),

\[
\Delta \lambda_{i+1} = \Delta \lambda_i - \frac{f_{n+1}[\Delta \lambda_i]}{\frac{df_{n+1}}{d(\Delta \lambda)}|_{i}}
\]
Based on the solution $\Delta \lambda_i$ of the previous iteration, we calculate the partial derivatives

$$\frac{\partial f_{n+1}}{\partial (\lambda_i)} = \frac{\partial \sigma_{n+1}}{\partial (\lambda_i)} - \frac{\partial k_{n+1}}{\partial (\lambda_i)}$$

(35)

using the chain rule. Defining $\xi_i = \sigma - \alpha$, the first term of (35) reads

$$\frac{\partial \sigma_{n+1}}{\partial (\lambda_i)} = \frac{\partial \xi}{\partial (\lambda_i)} = (\frac{\partial \sigma_{n+1}}{\partial (\lambda_i)} - \frac{\partial \alpha_{n+1}}{\partial (\lambda_i)})$$

(36)

with

$$\frac{\partial \xi}{\partial (\lambda_i)}_{n+1} = P \left( \frac{\xi_{n+1}}{\sigma_{n+1}} \right)$$

(37)

$$\frac{\partial \sigma_{n+1}}{\partial (\lambda_i)} = \frac{\partial \xi}{\partial (\lambda_i)}_{n+1} = -C \frac{\partial \xi}{\partial (\lambda_i)}_{n+1}$$

(38)

$$\frac{\partial \alpha_{n+1}}{\partial (\lambda_i)} = \frac{\partial \xi}{\partial (\lambda_i)}_{n+1} = \frac{2}{3} C \frac{\partial \xi}{\partial (\lambda_i)}_{n+1} - c_n \alpha_{n+1}$$

(39)

The second term of (35) reads

$$\frac{\partial k_{n+1}}{\partial (\lambda_i)} = AH \exp \left[ -A \gamma_{n+1}^\mu \right] \frac{\partial \gamma_{n+1}^m}{\partial (\lambda_i)}_{n+1} + H \frac{\partial \gamma_i}{\partial (\lambda_i)}_{n+1}$$

(40)

with

$$\frac{\partial \gamma_{n+1}^m}{\partial (\lambda_i)}_{n+1} = \sqrt{B \frac{\partial \xi}{\partial (\lambda_i)}_{n+1} - \frac{\partial \xi}{\partial (\lambda_i)}_{n+1}}$$

(41)

$$B = (e_1 \otimes e_1 + e_2 \otimes e_2) + \frac{1}{2} (e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{1}{2} I$$

(42)

**Fig. 12.** Deformation history for plane strain tension along the material 90° direction during 360° rigid body rotation of a single shell element (S4R), with local coordinate system shown in white.
and

\[
\frac{\partial \chi}{\partial \lambda_{n+1}} = (\lambda_{\text{max}} - \chi_{n+1}) m D (D \varepsilon_{\text{vM},n+1}^p)^{m-1} \frac{\partial \bar{\sigma}_{\text{vM}}}{\partial \lambda_{n+1}}
\]

(43)

with the stress-state dependent function

\[
D = D_0 + a_p \eta [\sigma_{n+1}] + a_u \bar{\eta} [\sigma_{n+1}]
\]

(44)

After calculating the new solution \( \Delta \lambda_{n+1} \) using (34), we update the internal state variables

\[
\alpha_{n+1} = \alpha_n + \left. \frac{\partial \alpha}{\partial \lambda} \right|_{n+1} (\Delta \lambda)
\]

(45)

\[
\chi_{n+1} = \chi_n + \left. \frac{\partial \chi}{\partial \lambda} \right|_{n+1} (\Delta \lambda)
\]

(46)

and the equivalent plastic strains

\[
\varepsilon_{\text{vM},n+1}^p = \varepsilon_{\text{vM},n}^p + \left. \frac{\partial \varepsilon_{\text{vM}}^p}{\partial \lambda} \right|_{n+1} (\Delta \lambda)
\]

(47)

\[
\bar{\varepsilon}_{n+1} = \bar{\varepsilon}_n + \Delta \lambda
\]

(48)

Subsequently, the stresses are updated

\[
\sigma_{n+1} = \sigma_n - C \left. \frac{\partial \sigma}{\partial \lambda} \right|_{n+1} (\Delta \lambda)
\]

(49)

\[
\xi_{n+1} = \sigma_{n+1} - \alpha_{n+1}
\]

(50)

\[
\bar{\sigma}_{n+1} = \sqrt{P \bar{\xi}_{n+1} \bar{\xi}_{n+1}}
\]

(51)

Fig. 13. Stress versus time curves (top) and strain versus time (bottom) for plane strain tension in the cross-rolling direction (solid lines) and an in-plane rigid body rotation during plane strain tension along the cross-rolling direction (symbols).
\[ k_{n+1} = k_0 + H_0 \left\{ 1 - \exp\left[ -\Delta \tau_{e,n} \right] \right\} + H_2 T_{n+1} \]

before evaluating
\[ f_{n+1} = f_{n+1} - k_{n+1} \]

The iterative procedure is continued until the criterion
\[ |f_{n+1}| \leq TOL \]

is met for \( TOL = 10^{-2} \) MPa.

To demonstrate the objectivity of the above time integration scheme, we apply plane strain tension along with a rigid body rotation to a single shell element. For plane strain tension along the cross-rolling direction along with in-plane rigid body rotation, the current position \( \mathbf{x} \) for a point within a single shell element is described by
\[ \mathbf{x}(t) = \mathbf{L}(t) \mathbf{X} + \mathbf{e}_0 [\mathbf{e}_2] - \mathbf{X} \]

with the rotation
\[ \mathbf{L}(t) = (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) \cos(\hat{\theta}t) + (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) \sin(\hat{\theta}t), \]

and the position vector \( \mathbf{X} \) in the reference configuration. 100,000 explicit time steps are performed to reach a maximum strain of \( \varepsilon_{\text{eq}} = 0.2 \) while rotating the shell element by \( 2\pi \) at a constant loading velocity (\( \dot{\theta} = \text{const.}, \varepsilon_{\text{eq}} = \text{const.} \)). Fig. 12 shows selected intermediate configurations of the deforming shell element. Fig. 13 shows the stress and strain evolutions in the single shell element using the user-defined plasticity model. The symbols in each figure give the curves for an element undergoing plane strain tension along with rigid body rotation. The data from this deformation is compared with the evolution of stress and strain for an element that undergoes plane strain tension with zero rigid body rotation (solid lines). As expected, we observe the curves are identical, which illustrates the objectivity of the proposed time integration scheme.

References


