Anisotropic Hosford–Coulomb fracture initiation model: Theory and application

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Abstract

An anisotropic extension of the Hosford–Coulomb localization criterion is obtained through the linear transformation of the stress tensor argument. Unlike for isotropic materials where the stress state is characterized through the stress triaxiality and Lode parameter, the normalized Cauchy stress tensor is used to describe the stress state in an anisotropic solid. Based on experiments on extruded aluminum 6260-T6 covering stress states from pure shear to equi-biaxial tension for different material orientations, it is shown that this phenomenological and uncoupled model is capable to provide reasonable engineering approximations of the strains and displacements to fracture for thirteen different loading conditions.

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1. Introduction

It is important to accurately predict the ductile fracture of metallic materials for lightweight structure design. Damage evolution within the material is usually considered to be the reason for fracture initiation. Gurson [29] porous plasticity model describes the growth of voids and its effect on the plastic behavior. The model was later modified by taking the void nucleation and coalescence into consideration [55,56]. Many recent efforts have been focused on improving the prediction capability of porous plasticity models for shear-dominated softening and failure at low stress triaxiality [21,32,47,60].

Another approach for modeling ductile fracture is the Continuum Damage Mechanics (CDM) framework where a phenomenological scalar damage variable is derived from a thermodynamic dissipation potential [10,11]; Chaboche, 1988; Lemaitre, 1985).

Besides the Gurson-like and CDM models, there are numerous so-called uncoupled phenomenological and empirical models proposed for predicting ductile fracture initiation, in which the elastic and plastic behavior is assumed to be unaffected by the evolution of damage. As summarized by Wierzbicki et al. [59], phenomenological models can be simply strain-based [31,37], stress-based [4,9], or embedded into a damage indicator framework (e.g. [33,19]). A comparative study on eight damage indicator models were carried out by Bao and Wierzbicki [3] with the weighting functions of the models derived respectively from the work of McClintock [43], Rice and Tracey [51], Leroy et al. [38], Clift et al. [18] and Oh et al. [49]. It is common to utilize the stress triaxiality as the only stress state parameter in damage indicator models [3,48].
Recently, some more general phenomenological models, which take the effect of both the stress triaxiality and the Lode angle parameter into consideration, were proposed by Bai and Wierzbicki [1,2], Coppola et al. [20], Gruben et al. [28], Lou et al. [41], Lou and Huh [40], and Voyiadis et al. [58], respectively. Ebnoether and Mohr [25] compared the original stress-based Mohr–Coulomb model and the damage indicator model derived from the Mohr–Coulomb model with their predicted results of the ductile fracture of low carbon steel sheets; it was found that the damage indicator model showed better accuracy in simulating the stamping experiment where the loading history was highly non-linear. Comparison of the predictive capabilities of uncoupled phenomenological, Gurson-like and CDM models were performed by Dunand and Mohr [24] and Li et al. [39].

Enhanced Gurson-type models have been developed accounting for matrix anisotropy [6,7], spheroidal voids [26,27,12], a combination of both [46], and kinematic hardening (e.g. [44], Leblond et al., 1995). The effects of the void shape and void distribution were also covered in the models proposed by Pardoen [50], Pardoen and Hutchinson (2003) and Steglich et al. [53]. Steglich et al. [54] developed another Gurson-like model incorporating the direction-dependent void growth. A summary of the anisotropic yield growth and coalescence models can be found in the review by Benzerga and Leblond [8]. By expressing the scalar phenomenological damage as a tensorial parameter, a series of studies on modeling anisotropic damage and ductile fracture were performed within the framework of nonlinear continuum damage mechanics [13–17]. Similar approaches with the variable of a damage tensor introduced into anisotropic CDM models can be also found in the work by Hammi and Horstemeyer [30], Solanki et al. [52] and Voyiadis and Dorgan [57].

Most of the previous studies on modeling anisotropic ductile fracture follow the approach of either the Gurson-like or the CDM model. For modeling the deformation-induced anisotropic fracture response of a Ti–6Al–4V alloy, Khan and Liu [35] proposed an uncoupled stress-based criterion, which was extended from an isotropic one based on the magnitude of stress vector [34] with a modified Hill anisotropic function [36]. Luo et al. [42] developed an anisotropic damage indicator model for describing the fracture behavior of the extruded sheets of 6260-T6 aluminum alloy, using the isotropic Modified Mohr–Coulomb stress state weighting function [2] and the von Mises equivalent plastic strain definition after linear transformation of the plastic strain tensor.

The present work proposes an anisotropic extension of the recently proposed micro-mechanically motivated Hosford–Coulomb fracture initiation model [45]. Anisotropy is introduced into the originally isotropic formulation through the linear transformation of the stress tensor. After discussing its direction dependent features, it is demonstrated that the anisotropic Hosford–Coulomb model is able to provide reasonable predictions of the strain and displacement to fracture in experiments on highly anisotropic aluminum 6260-T6 extrusions subject to thirteen different nearly proportional loading paths in stress space.

2. Anisotropic plasticity model

2.1. Anisotropic yield function

The yield function is a 3D-extension of the anisotropic non-quadratic plane stress yield function Yld2000-2d of Barlat et al. [5]. It may be expressed in terms of an anisotropic equivalent stress measure, $\sigma_a$, and a deformation resistance $k$,

$$f(\sigma) = \sigma_a - k = 0.$$  \hspace{1cm} (1)

The equivalent stress is defined as an anisotropic function of the Cauchy stress tensor $\sigma$ in the material coordinate system,

$$\sigma_a[\sigma] = \frac{1}{2^{1/d}} (\phi'[\mathbf{s}'] + \phi''[\mathbf{s}''])^{\frac{1}{2}}$$  \hspace{1cm} (2)

with

$$\phi'[\mathbf{s}'] = \left[ (s_{11}^r - s_{22}^r)^2 + 4(s_{12}^r + s_{13}^r + s_{23}^r) \right]^\frac{d}{2}$$  \hspace{1cm} (3)

$$\phi''[\mathbf{s}'] = \left[ \frac{3}{2} (s_{11}^r - s_{22}^r)^2 + \frac{1}{2} \sqrt{ (s_{11}^r - s_{22}^r)^2 + 4(s_{12}^r + s_{13}^r + s_{23}^r)^2 } \right]^d$$

$$+ \left[ \frac{3}{2} (s_{11}^r - s_{22}^r)^2 - \frac{1}{2} \sqrt{ (s_{11}^r - s_{22}^r)^2 + 4(s_{12}^r + s_{13}^r + s_{23}^r)^2 } \right]^d$$  \hspace{1cm} (4)

and the linearly transformed deviatoric stress tensors $\mathbf{s}'$ and $\mathbf{s}''$. In vector notation,

$$\mathbf{s} = \{ \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13} \}$$  \hspace{1cm} (5)

$$\mathbf{s}' = \{ s_{11}', s_{22}', s_{12}', s_{23}', s_{13}' \}$$  \hspace{1cm} (6)

$$\mathbf{s}'' = \{ s_{11}'', s_{22}'', s_{12}'', s_{23}'', s_{13}'' \}$$  \hspace{1cm} (7)
the linear transformations read
\[ \dot{\mathbf{s}} = L' \dot{\mathbf{\sigma}} \] (8)
\[ \dot{\mathbf{s}}'' = L'' \dot{\mathbf{\sigma}} \] (9)
with the transformation matrices
\[ L' = \frac{1}{3} \begin{bmatrix}
2x_1 & -x_1 & -x_1 & 0 & 0 & 0 \\
-x_2 & 2x_2 & -x_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 3x_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 3x_9 & 0 \\
0 & 0 & 0 & 0 & 0 & 3x_{10}
\end{bmatrix} \] (10)
\[ L'' = \frac{1}{9} \begin{bmatrix}
-2x_3 + 2x_4 + 8x_5 - 2x_6 & -4x_4 - 4x_6 + x_3 - 4x_5 - 2x_6 & 0 & 0 & 0 \\
4x_3 - 4x_4 - 4x_5 + x_6 & -2x_3 + 8x_4 + 2x_5 - 2x_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 9x_8 \\
0 & 0 & 0 & 0 & 9x_11 \\
0 & 0 & 0 & 0 & 0 & 9x_{12}
\end{bmatrix} \] (11)
The anisotropic equivalent stress is therefore defined through the exponent \( d \) and the 12 transformation coefficients \( x_i \), \( i = 1, 2, \ldots, 12 \). As discussed by Dunand et al. [22], the above yield function reduces to the Yld2000-2d model when plane stress conditions prevail in the \((e_1, e_2)\)-plane (i.e. \( \sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \)).

2.2. Associated flow rule

The model assumes associated plastic flow,
\[ d\mathbf{e}_p = d\lambda \frac{\partial f}{\partial \mathbf{\sigma}} \] (12)
with the anisotropic equivalent plastic strain
\[ \bar{\mathbf{e}}_a = \int d\lambda. \] (13)
defined as work-conjugate to the anisotropic equivalent stress,
\[ \mathbf{\sigma} : (d\mathbf{e}_p) = \bar{\mathbf{\sigma}}_a (d\bar{\mathbf{e}}_a). \] (14)

2.3. Isotropic hardening

For the aluminum 6260-T6 alloy, self-similar (isotropic) hardening behavior is described through the mixed Swift–Voce law [e.g. [45]],
\[ k(\bar{\mathbf{e}}_a) = w A_3 (\bar{\mathbf{e}}_a + \bar{\mathbf{e}}_c)^{n_3} + (1 - w) \{ k_0 + Q (1 - e^{-\beta \bar{\mathbf{e}}_a}) \}. \] (15)
with the Swift parameters \( \{A_3, \bar{\mathbf{e}}_c, n_3\} \), the Voce parameters \( \{k_0, Q, \beta\} \), and the weighting factor \( w \).

3. Isotropic Hosford–Coulomb fracture initiation model

The formulation of the isotropic Hosford–Coulomb model as proposed by Mohr and Marcadet [45] is briefly recalled. We adopt a slightly different notation in view of developing its anisotropic extension in Section 4. Note that the argument of a function is always enclosed into square brackets, while parentheses are used to define the order of mathematical operations.

3.1. Starting point: isotropic Hosford–Coulomb criterion in stress space

Defining the Hosford equivalent stress as
\[ \bar{\sigma}_{HF} = \left\{ \frac{1}{2} \left( (\sigma_I - \sigma_II)^a + (\sigma_I - \sigma_{III})^a + (\sigma_{II} - \sigma_{III})^a \right) \right\}^{\frac{1}{2}} \] (16)
Mohr and Marcadet [45] proposed the Hosford–Coulomb criterion
\[ g_{HC}[\mathbf{\sigma}] := \bar{\sigma}_{HF} + C(\sigma_I + \sigma_{III}) = \beta \] (17)
to describe the onset of shear and normal localization in metals under proportional loading. The model features three parameters, with \( \alpha \) denoting the Hosford exponent, the cohesion \( \beta \) and the friction coefficient \( c \). For \( \alpha = 1 \), the model reduces to a Mohr–Coulomb model which is independent of the intermediate principal stress. For \( \alpha \neq 1 \), the Hosford–Coulomb model depends on all three principal stresses.

### 3.2. Isotropic fracture initiation model

For isotropic materials, the stress state is typically characterized through the stress triaxiality,

\[
\eta = \frac{\sigma_m}{\sigma} \tag{18}
\]

and the Lode angle parameter \( \vartheta \),

\[
\vartheta = 1 - \frac{2}{\pi} \arccos \left( \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \right) \tag{19}
\]

For anisotropic materials, the orientation of the stress tensor with respect to the material directions must also be taken into account (e.g. three Euler angles) in addition to the stress triaxiality and the Lode parameter. In other words, five parameters are needed to describe the stress state for anisotropic materials. As an alternative to working with Euler angles, we make use of the stress triaxiality in the material coordinate system as normalized by the von Mises equivalent stress, \( \sigma / \sigma \), to characterize the stress state. Note that \( \sigma / \sigma \) features five independent components only. In terms of the normalized stress tensor, criterion (17) reads,

\[
\sigma_{pr}^e = \frac{\beta}{g_{HC}[\sigma / \sigma]} \tag{20}
\]

with \( \sigma_{pr}^e \) denoting the von Mises equivalent stress at the onset of fracture. Introducing a power law to define a bijective mapping from equivalent stress to equivalent strain space,

\[
\sigma_{pr}^e = h \left( \frac{\sigma_{pr}^e}{\sigma} \right) = A \left( \frac{\sigma_{pr}^e}{\sigma} \right)^n \tag{21}
\]

leads to an expression of the equivalent plastic strain to fracture as a function of the stress state:

\[
\varepsilon_{pr}^f[\sigma / \sigma] = h^{-1} \left[ \frac{\sigma_{pr}^e}{\sigma} \right] = \left( \frac{\beta}{A} \right)^{1/n} \left( \frac{1}{g_{HC}[\sigma / \sigma]} \right)^{1/n} \tag{22}
\]

In the above form the coefficients \( \beta \) and \( A \) both control the magnitude of the strain to fracture. We therefore substitute the multiplier \( (\beta / A)^{1/n} \) through a single coefficient. Moreover, defining the new coefficient \( b \) such that it corresponds to the strain to fracture for uniaxial tension results in the final form of the Hosford–Coulomb fracture initiation model for proportional loading:

\[
\varepsilon_{pr}^f[\sigma / \sigma] = b \left( \frac{1 + c}{g_{HC}[\sigma / \sigma]} \right)^{1/n} \tag{23}
\]

Note that we introduced the superscript \( 'pr' \) to indicate that the above expressions are applied to predict the onset of fracture under proportional loading only. For non-proportional loading, the integral form

\[
\int_0^\varepsilon \frac{d\varepsilon}{\varepsilon_{pr}^f[\sigma / \sigma]} = 1 \tag{24}
\]

is used, where the expression for the strain to fracture for proportional loading serves as weighting function. Note that the result \( \varepsilon_f = \varepsilon_{pr}^f[\sigma / \sigma] \) is obtained from (22) in the case of proportional loading.

### 3.3. Illustration

The above fracture initiation model features four parameters: \( a, b, c \) and \( n \). In view of developing an anisotropic model, the above equations have been expressed in terms of the normalized stress tensor \( \sigma / \sigma \). However, if the tensor \( \sigma / \sigma \) is used as the argument of an isotropic function \( f_{iso}[\sigma / \sigma] \), then there exists a function \( f_{iso}[\eta, \theta] \) such that \( f_{iso}[\sigma / \sigma] = f_{iso}[\eta, \theta] \). In the case of the isotropic Hosford–Coulomb criterion (21), we have

\[
g_{HC}[\sigma / \sigma] = g_{HC}[\eta, \theta] = \left\{ \frac{1}{2} (J_1 - f_2)^a + (f_2 - f_3)^a + (f_1 - f_3)^a \right\}^{1/2} + c(2\eta + f_1 + f_3) \tag{25}
\]
with the Lode angle parameter dependent trigonometric functions
\[ f_1[\theta] = \frac{2}{3} \cos\left[\frac{\pi}{6}(1 - \theta)\right], \quad f_2[\theta] = \frac{2}{3} \cos\left[\frac{\pi}{6}(3 + \theta)\right] \quad \text{and} \quad f_3[\theta] = -\frac{2}{3} \cos\left[\frac{\pi}{6}(1 + \theta)\right]. \]

Fig. 1a shows a plot of the isotropic Hosford–Coulomb fracture initiation model for proportional loading. The strain to fracture is a monotonically decreasing function of the stress triaxiality and a convex non-symmetric function of the Lode angle parameter. The corresponding strain to fracture for plane stress conditions (see black curve in Fig. 1a) is shown as a function of the stress triaxiality only in Fig. 1b. Note that the apparent non-monotonic dependence on the stress triaxiality is deceiving because of the underlying Lode angle effect caused by the functional relationship
\[ \theta = 1 - \frac{2}{\pi} \arccos\left[-\frac{27}{2} \eta\left(\eta^2 - \frac{1}{3}\right)\right] \quad \text{for} \quad -2/3 \leq \eta \leq 2/3 \]  

The effect of the model parameters \(a\) and \(c\) is illustrated in Fig. 2. Note that \(b\) has always been chosen such that the strain to fracture for uniaxial tension equals 0.3. Observe from Fig. 2a that the lower the Hosford exponent, the deeper the "biaxial tension valley" between uniaxial and equi-biaxial tension. The friction parameter \(c\) controls the stress triaxiality dependency; the higher \(c\), the greater the difference between the strain to fracture for pure shear (\(\eta = 0, \theta = 0\)) and plane...
strain tension ($\eta = 0.58, \dot{\varepsilon} = 0$). Fig. 3c. The effect of the parameter $n$ is similar to that of the exponent $a$. It does not affect the monotonicity of the fracture envelope, but can amplify ($n \rightarrow 0$) and de-amplify ($n \rightarrow 1$) the differences in the strain to fracture for different stress states (Fig. 2e). Irrespective of the choice of the model parameters, the strain to fracture for uniaxial...
and equi-biaxial tension are always identical. This is due to the fact that the Hosford equivalent stress is equal to the maximum principal stress for these two stress states. Furthermore, with the minimum principal stress being zero \( \sigma_{III} = 0 \) in both cases, the Hosford–Coulomb model (Eq. (17)) reduces to maximum principal stress criterion for uniaxial and equi-biaxial tension.

### 3.4. Underlying model in stress space (for proportional loading)

The above formulation deviates from that proposed by Mohr and Marcadet [45] because of the power law transformation (19) from the \( \{\sigma, \eta, \theta\} \) stress space to the \( \{\tilde{\epsilon}_p, \eta, \theta\} \) mixed strain–stress space, instead of using the isotropic hardening law (15). This deviation is necessary due to the fact that the bijective nature of the \( \sigma \) to \( \tilde{\epsilon}_p \) mapping defined by the isotropic hardening law is barely visible (i.e. almost the same stress is defined for different strains) in case of materials that feature pronounced saturation type of hardening, thereby making the model predictions sensitive to the numerical noise, in particular when using a single precision code.

However, it is emphasized that for proportional loading, using (21) is mathematically equivalent to using a localization criterion in stress space. The corresponding criterion in stress space is obtained by transforming the criterion from mixed stress–strain space \( \{\eta, \theta, \tilde{\epsilon}_p\} \) to the modified Haigh–Westergaard space \( \{\eta, \theta, \sigma_p^m\} \) using the material’s isotropic hardening law (15). For a Levy–von Mises material, we have \( \tilde{\epsilon}_p = \tilde{\epsilon}_a \) and thus \( \sigma_p^m = k \tilde{\epsilon}_p^m \); the resulting criterion in stress space would therefore read

\[
\sigma_j^m = k \left[ \tilde{\epsilon}_p^m / \sigma \right] = k \left[ h^{-1} \left( \frac{\beta}{g_{HC} \sigma} \right) \right] \quad \text{with } \beta = A(1 + c) b^n
\]  

The corresponding criterion in stress space is the Hosford–Coulomb (HC) criterion given by Eq. (1) only if \( k\tilde{\epsilon}_p = h\tilde{\epsilon}_p \), i.e. only if the material’s hardening response follows a power law of exponent \( n \). Otherwise a modified Hosford–Coulomb (mHC) criterion is obtained in stress space.
Fig. 2f serves as an illustration of the differences between the original and modified Hosford–Coulomb model. For example, assuming that the material’s hardening law is given by \( k \dot{\varepsilon}_p = A \dot{\varepsilon}_p^0 \), the red\(^1\) envelope with \( n = 0.1 \) would represent the original HC model in stress space (for \( a = 1.3 \) and \( c = 0.05 \)). The other three envelopes \((n = 0.05, n = 0.2 \) and \( n = 0.4 \)) would then represent a modified HC criterion in stress space. Clearly, the differences are small in stress space (Fig. 2f), but these can be very significant in the mixed strain–stress space (Fig. 2e).

4. Anisotropic Hosford–Coulomb fracture initiation model

In the context of phenomenological modeling, a first step towards the formulation of an anisotropic fracture initiation model is substituting the von Mises equivalent plastic strain \( \dot{\varepsilon}_p \) by its anisotropic counterpart \( \dot{\varepsilon}_a \). The ductile fracture model then reads

\[
\int_0^{\dot{\varepsilon}_a} \frac{d\dot{\varepsilon}_a}{\dot{\varepsilon}_a^p [\dot{\varepsilon}_a/\dot{\varepsilon}_a]} = 1 \quad \text{with} \quad \dot{\varepsilon}_a^p [\dot{\varepsilon}_a/\dot{\varepsilon}_a] = b \left( \frac{1 + c}{g_{HC}[\sigma/\sigma]} \right)^{1/2}.
\]

For ease of notation, we use the Hosford–Coulomb function \( g_{HC}[\sigma] \) with the stress vector \( \sigma \) as argument, \( g_{HC}[\sigma] = g_{HC}[\sigma] \). In Eq. (28), the entire anisotropy of the fracture model is inherited from the plasticity model. As shown by Luo (2012), this approach is not sufficient to account for the direction dependency of ductile fracture. The concept of linear stress tensor transformation has been successfully used in the past to derive anisotropic yield criteria from isotropic functions (Karafillis and Boyce, 1993, [5], Besson and Bron, 2004). Here, the same idea is exploited to introduce anisotropic stress state dependency into the fracture initiation model. With \( M \) denoting a \( 6 \times 6 \) matrix, the weighting function of the anisotropic Hosford–Coulomb fracture initiation model is written as

\[
\int_0^{\dot{\varepsilon}_a} \frac{d\dot{\varepsilon}_a}{\dot{\varepsilon}_a^p [\dot{\varepsilon}_a/\dot{\varepsilon}_a]} = \text{with} \quad \dot{\varepsilon}_a^p [\dot{\varepsilon}_a/\dot{\varepsilon}_a] = b \left( \frac{1 + c}{g_{HC}[M \sigma/\sigma]} \right)^{1/2}.
\]

In view of modeling thin-walled structures, we chose a linear transformation matrix of the form

\[
\mathbf{M} = \begin{bmatrix}
1 & M_{12} & 0 & 0 & 0 & 0 \\
0 & M_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

with the transformation coefficients \( M_{12}, M_{22} \) and \( M_{44} \). In close analogy with the discussion of the isotropic model (Section 3.4), it can be demonstrated that the proposed anisotropic model reduces to a criterion in stress space for proportional loading which is given through the implicit form

\[
\dot{\varepsilon}_a^p = k \left[ b \left( \frac{1 + c}{g_{HC}[M \sigma/\sigma]} \right)^{1/2} \right]
\]

with \( \dot{\varepsilon}_a^p \) denoting the anisotropic equivalent stress (defined by Eq. (2)) at the instant of onset of fracture.

4.1. Illustration

According to the proposed model, the equivalent plastic strain to fracture varies as a function of the loading path due to a combination of two effects:

1. Stress state dependency of the strain to fracture, i.e. the strain to fracture varies as a function of the stress triaxiality and the Lode angle parameter which are both isotropic functions of the normalized stress tensor \( \sigma/\sigma \);

2. Anisotropy of the fracture model which is introduced through the linear transformation \( \mathbf{M} \) of the stress tensor;

For illustration purposes, we consider the model response for proportional loading only. Firstly, we consider the model response for biaxial tension in the \( (\sigma_{11}, \sigma_{22}) \)-plane. Given that the stress state depends on the stress ratio \( \sigma_{22}/\sigma_{11} \) only, we present the results in polar plots \( (\varphi, r) \) with the polar coordinates \( \varphi = \arctan(\sigma_{22}/\sigma_{11}) \) and \( r = \dot{\varepsilon}_a^p \). The default parameters for the plots shown in Figs. 3–5 are: \( a = 1.3, b = 0.9, c = 0, n = 0.2, M_{12} = 0, M_{22} = 1 \) and \( M_{44} = 1 \).

\(^1\) For interpretation of color in Figs. 2 and 12, the reader is referred to the web version of this article.
As shown previously, the parameter $a$ influences the model’s stress state sensitivity. The lower $a$, the greater the variation in the strain to fracture as a function of the loading path. All curves shown in Fig. 3a correspond to isotropic fracture envelopes. Note that for $a = 2$ the strain to fracture is both stress state and loading direction independent.

The influence of the parameter $n$ is similar to that of $a$, i.e. for $a \neq 2$, it amplifies the stress state dependency for $n \to 0$ (Fig. 3c).

For the anisotropic model, $b$ is the strain to fracture for uniaxial tension along the 1-direction. For all loading paths, the strain to fracture is proportional $b$ (Fig. 4a).

The parameters $M_{12}$ and $M_{22}$ control the anisotropic distortion of the fracture envelope. The strain to fracture for uniaxial tension along the 2-direction decreases as a function of $M_{22}$ (Fig. 5a); it increases as a function of $M_{12}$ (Fig. 5c).

The effect of the remaining parameters can be best seen in the plane $(\sigma_{12}, \sigma_d)$ with $\sigma_{11} = \sigma_d/\sqrt{2}$ and $\sigma_{22} = \sigma_d/\sqrt{2}$. This particular plane is chosen as the condition $\sigma_{11} = \sigma_{22}$ is often met in experiments along the 45$^\circ$-direction.

- The parameter $M_{44}$ results in an expansion/contraction along the in-plane shear direction (Fig. 5f).
- The parameter $c$ has a similar effect on the failure envelope (Fig. 4d). The key difference as compared to $M_{44}$ is that $c$ has an isotropic effect and also changes the envelope in the $(\sigma_{11}, \sigma_{22})$-plane (compare Figs. 4c and 5e).

### 5. Application

The experiments performed by Dunand et al. (2013) and Luo et al. (2013) will serve as basis for calibrating and validating the anisotropic Hosford–Coulomb model.

In a first step, we revisit the calibration of the plasticity model before analyzing the fracture experiments.

#### 5.1. Material and experiments

The experimental data used in this study had been obtained from experiments on aluminum 6260-T6 specimens. All specimens had been extracted from the 2 mm thick walls of an extruded automotive part. As emphasized by Luo et al.
(2013), all specimens came from the same location within the extruded profiles. Following the discussion of Rousselier et al. (2013), only the results for full thickness specimens are considered in this study. The yield stress and Lankford ratios (slope of the logarithmic plastic width strain versus plastic thickness strain plot) as determined from uniaxial tensile experiments for seven different in-plane orientations (15° increments) are shown in Fig. 6a. The yield stress variations do not exceed 5%, while the Lankford ratios vary significantly as a function of the specimen orientation, from \( r_{30} = 0.25 \) (minimum) to \( r_{90} = 1.0 \) (maximum). Note that an \( r \)-value below 1 indicates that the thickness reduction is more pronounced than the reduction in width under uniaxial tension.

The fracture specimen geometries are shown in Fig. 7. The experimental program includes notched tensile specimens with a minimum gage section width of 10 mm and three different notch radii: \( R = 5 \) mm (NT5-specimen), \( R = 10 \) mm
Fig. 6. Anisotropic plasticity of Al 6260-T6: (a) Lankford and yield stress ratios as a function of the material orientation, (b) stress–strain curve defining the self-similar (isotropic) hardening.

Fig. 7. Fracture specimen geometries. The displacement measured by a 30 mm (20 mm) long virtual axial strain gage is reported for NT (CH) specimens. Note that all specimens feature 35 × 10 mm large gripping areas at the top and bottom.
(NT10-specimen) and \( R = 20 \) mm (NT20-specimen). In addition, 20 mm wide tensile specimens with a \( R = 5 \) mm radius central hole are tested (CH-specimens). Each type of specimen has been extracted for three different material orientations with respect to the tensile axis. In the 0°-specimens (NT5-0, NT10-0, NT20-0 and CH-0), the tensile direction coincides with the tensile axis. Analogously, 90°-specimens (NT5-90, NT10-90, NT20-90 and CH-90) with the tensile axis being parallel to the transverse direction, and 45°-specimens (NT5-45, NT10-45, NT20-45 and CH-45) along the diagonal direction have been extracted from the extruded profiles through wire EDM cutting. In addition to these 12 tensile fracture experiments, we also include the result from a punch test on a disc specimen with a 45 mm diameter punch. All experiments had been performed under static loading conditions at a strain rate of less than \( 10^{-2} \) s\(^{-1}\).

Figs. 8 and 9 provide a summary of all measured force–displacement curves (solid dots). Each curve represents the average of two experiments. Note that the scatter in the experimental results for repeated experiments was small, i.e. the force displacement curves lied on top of each other with variations in the displacement to fracture of less than 3% \([42]\). The reported displacements correspond to the relative displacement of two points positioned on the respective upper and lower specimen shoulders at an initial distance of 30 mm and 20 mm for the NT and CH specimens, respectively; it has been measured using planar digital image correlation.

5.2. Finite element models

A numerical simulation is performed of each fracture experiment to determine the so-called loading path to fracture, i.e. the evolution of the anisotropic equivalent plastic strain as a function of the stress state up to the instant of fracture initiation. We follow the recommendations of Luo et al. \([42]\) and use eight first-order solid elements with reduced integration (element C3D8R from the Abaqus library) along half the specimen thickness. Details of the meshes become visible when zooming into the contour plots shown in Figs. 8 and 9 (of the electronic version of this paper).

The finite element meshes for the tensile specimens discretize the specimen geometry between the points of displacement measurement. The displacement loading is then applied over at least 100,000 explicit time steps up to the experimentally measured displacement to fracture (assuming that fracture initiates with the first drop in force in the experiments). A friction coefficient of 0.05 has been used in the simulations of the punch experiment. Due to the symmetry of the mechanical problem, only one eighth of the tensile specimens is modeled and one quarter of the punch specimen. A user material subroutine is used to simulate the anisotropic material response as outlined by the constitutive equations above.

5.3. Plasticity model calibration and validation

The same parameters \( x_i \) as those identified by Dunand et al. \([22]\) are chosen to describe the shape of the anisotropic yield surface (Table 1). This set of parameters provides an accurate description of the Lankford ratio variations (compare solid line with solid dots in Fig. 6a). The basis for their identification were the yield stresses and Lankford ratios as determined from tension experiments along the 0°, 45° and 90° directions. In addition, the yield stresses from shear experiments with two distinct specimen orientations had been used. In an attempt to improve the model response in the post-necking range (as compared to Dunand et al. \([22]\)), the isotropic strain hardening function has been recalibrated using Eq. (15) instead of a Swift law only. The parameters \((A, e_0, n)\) are found from a fit of the Swift law to the uniaxial stress–strain curves (which are valid up to a strain of about 0.05). Analogously, the parameters \((k_0, Q, \beta)\) are found from a fit of the Voce law. As differences between the Swift and Voce laws greater than 5% in stress only become apparent for strains greater than 0.1, we chose the central hole tension experiment along the extrusion direction (where equivalent plastic strains above 0.5 can be reached) to identify the weighting factor \(w\) through inverse analysis. For this, we coupled the Abaqus solver with the optimization toolbox of Matlab and made use of a Nelder-Mead algorithm (function fminsearch) to minimize the difference between the simulated and measured force–displacement curves. The final parameters are summarized in Table 2 and the corresponding stress–strain curve is shown in Fig. 6b. The predictions of the force–displacement curves for all other experiments are shown as solid black lines in Figs. 8 and 9. Reasonable agreement is observed for all specimens, with the largest discrepancies for the NT5 experiments.

5.4. Loading paths to fracture

The computed contour plots of the equivalent plastic strain within the specimen gage section at the instant of onset of fracture initiation are shown for each specimen in Figs. 8 and 9. The loading paths to fracture are extracted at the location where the highest equivalent plastic strain is observed. Recall that the stress state for an anisotropic material is defined through the normalized tensor \(\overline{\sigma}\). The evolution of the entire Cauchy stress tensor is therefore extracted in addition to the anisotropic equivalent plastic strain. The normalized stress tensor has five independent components. The complete loading path to fracture therefore describes an evolution in a six-dimensional space. However, for visualization purposes, we only show two-dimensional projections of these loading paths:

- Fig. 10a shows the projections of the loading paths onto the \((\sigma_{11}, \sigma_{22})\)-plane. Neglecting the contributions of the other stress tensor components is a reasonable assumption for the punch (PU) and tensile experiments (NT and CH) along the 0°- and 90°-directions.
Fig. 8. Force–displacement curves for notched tension as measured experimentally (solid dots) and extracted from FE analysis. The scale bar for the contour plots of the equivalent plastic strain distribution near the specimen centers is included in Fig. 9.
Fig. 9. Force–displacement curves for (a–c) tension with a central hole and (d) punch testing as measured experimentally (solid dots) and extracted from FE analysis.

Table 1
Calibrated coefficients for linear transformation of stress tensor for the plasticity model.

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$a_6$</th>
<th>$a_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.901</td>
<td>1.042</td>
<td>0.658</td>
<td>0.933</td>
<td>1.044</td>
<td>1.021</td>
<td>0.754</td>
</tr>
</tbody>
</table>

Table 2
Calibrated parameters for the isotropic hardening law.

<table>
<thead>
<tr>
<th>$A$ (MPa)</th>
<th>$\varepsilon_0$ (–)</th>
<th>$n$ (–)</th>
<th>$k_0$ (MPa)</th>
<th>$Q$ (MPa)</th>
<th>$\beta$ (–)</th>
<th>$w$ (–)</th>
</tr>
</thead>
<tbody>
<tr>
<td>325</td>
<td>0.007</td>
<td>0.1</td>
<td>205.2</td>
<td>48.4</td>
<td>23.7</td>
<td>0.054</td>
</tr>
</tbody>
</table>
Fig. 10 shows the projection of selected loading paths on the diagonal \((\sigma_{11}, \sigma_{12})\)-plane. Note that the shear stress component plays an important role in the tensile experiments on 45\(^\circ\)-specimens.

Due to the particular choice of coordinates, the results for an isotropic material would be symmetric with respect to the bisector line \((\sigma_{11} = \sigma_{22})\) in Fig. 10a. However, as demonstrated by the comparison of the loading paths for NT-90 and NT-0, and the comparison of the loadings paths for CH-90 and CH-0 demonstrate, a very strong anisotropy is observed in the material's fracture response. For example, the anisotropic equivalent plastic strain to fracture for uniaxial tension along the 0\(^\circ\)-direction \((e_{\text{pf}}^0 = 0.86)\) is about 90\% higher than that for the 90\(^\circ\)-direction \((e_{\text{pf}}^0 = 0.45)\).

5.5. Fracture initiation model calibration and validation

The parameters \(\{a, b, c, n\}\) and \(\{M_{22}, M_{44}, M_{12}\}\) describing the respective isotropic and anisotropic parts of the Hosford-Coulomb fracture initiation model are identified through inverse analysis:

- **Step1:** Identification of seed parameters for \(\{a, b, c, n\}\) assuming an isotropic fracture response. We impose \(M_{22} = M_{44} = 1\) and \(M_{12} = 0\) and run a first round of identification using the results for 0\(^\circ\)-specimens only (NT5-0, NT10-0, NT20-0, CH-0, PU).
Step 2: Full identification of all parameters \(a, b, c, n, M_{12}, M_{22}, M_{44}\) using the results from eight experiments: CH-0, CH-45, CH-90, NT10-0, NT10-45, NT10-90 and PU. The optimization is performed using a derivative-free simplex algorithm (Matlab) which minimizes the difference between the strains to fracture predicted by Eq. (29) with those measured experimentally.

The final set of parameters obtained after more than 150 iterations is given in Table 3. The model predictions of the strain to fracture are highlighted as solid dots in Fig. 10a and b. The comparison of the end points of the solid loading paths (experimental strain to fracture) with the corresponding solid dots shows the accuracy of the model predictions for all experiments. The maximum relative difference of 17% is observed for the NT5-90 experiment.

Another way of validating the model predictions is to compare the computed displacements to fracture with those measured experimentally. Fig. 11 shows this comparison demonstrating good agreement with a maximum relative difference of 7.3% for the CH-90 experiment. A \((\sigma_{11}, \sigma_{22})\) polar plot of the identified fracture envelope for proportional loading is shown in Fig. 12 (black solid curve) next to the isotropic envelopes for \(c = 0\) (blue dots) and \(c = 0.1\) (red dots). The calibrated envelope exhibits a pronounced compression–tension asymmetry (strain to fracture for uniaxial compression about 2.5 times higher than that for uniaxial tension) which is due to the normal stress effect that is introduced through the non-zero friction coefficient \(c\). At the same time, the linear transformation of the stress tensor argument ensures that the strain to fracture for uniaxial tension along the extrusion direction is more than twice as large as that for the transverse direction.

5.6. Comment on anisotropic fracture initiation model by Luo et al. [42]

The proposed model features the same number of parameters as the anisotropic MMC model proposed by Luo et al. [42]. The comparison of the model predictions (red and gray columns in Fig. 11) demonstrates that both models can fit the present experimental data equally well. However, the modeling approach is fundamentally different: the anisotropic MMC model is based on the linear transformation of the plastic strain tensor, while the anisotropic HC model is based on the linear transformation of the stress tensor. The latter approach is pursued here for conceptual reasons. The underlying fracture initiation mechanism is shear localization at the microscale. It has been demonstrated by Dunand and Mohr [23] through localization analysis that the onset of shear and normal localization in a Levy–von Mises solid under proportional loading can be described through a Hosford–Coulomb type of criterion in stress space. It is thus expected that the onset of shear localization

| Calibrated parameters for the anisotropic Hosford–Coulomb fracture initiation model. | Table 3 |
|---|---|---|---|---|---|---|---|
| \(M_{12}\) | \(M_{22}\) | \(M_{44}\) | \(a\) | \(b\) | \(c\) | \(n\) |
| \(-0.086\) | 1.182 | 1.114 | 1.035 | 1.015 | 0.1017 | 0.223 |

Fig. 11. Comparison of the displacement to fracture obtained from experiments (blue bars) and predicted by the proposed anisotropic Hosford–Coulomb model (red columns) and the anisotropic Modified Mohr–Coulomb model (gray columns). For the HC model, the maximum relative difference (7.3%) is observed for tension of a specimen with a central hole along the transverse direction (CH-90). For the punch loading (not shown), the HC simulation overestimates the displacement to fracture by 5.2%. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
(and hence the onset of fracture) in an anisotropic metal can be described through an anisotropic criterion in stress space. Consequently, a linear transformation of the stress tensor to account for anisotropy appeared to be conceptually more sound than transforming the strain tensor. A clear assessment of the predictive capabilities of these modeling approaches will require either more experimental data or results from localization analysis.

6. Conclusions

An anisotropic extension of the Hosford–Coulomb localization criterion is proposed to provide a phenomenological model for predicting the initiation of fracture in large scale simulations. Instead of using isotropic stress state measures (such as the stress triaxiality and the Lode angle parameter) along with the three principal stress orientations with respect to the material directions, the Cauchy stress tensor as normalized by the von Mises equivalent stress is directly used to describe the stress state in an anisotropic material. Starting with the formulation of the isotropic Hosford–Coulomb model in terms of the normalized Cauchy stress tensor, the anisotropic extension is developed through the linear transformation of the stress tensor argument. The influence of the seven model parameters is discussed in detail before their identification for extruded aluminum 6260-T6. The identification procedure involves the detailed finite element analysis of fracture experiments for thirteen different stress states (Luo et al., 2013) to extract the loading paths to fracture. The final set of model parameters is obtained though computational optimization. It is shown that the model is able provide a satisfactory engineering approximation of the observed fracture strains which ranged from 0.22 for notched tension along the transverse direction to values as high as 0.88 for uniaxial tension along the extrusion direction.

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References
